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New classes of analytic solutions of the two-level problem

A M Ishkhanyan

Engineering Center of the Armenian National Academy of Sciences, Ashtarak-2, 378410 Armenia

E-mail: artur@ec.sci.am

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Abstract. A simple technique to find analytic solutions of the time-dependent Schrödinger equations is introduced. A number of three-parametric classes of the two-level problem, integrable in terms of the Gauss hypergeometric function, including all presently known analytically solvable models are derived.

1. Introduction

The analytic solutions of the two-level problem have played a central role in studying a number of important physical phenomena in many branches of contemporary physics ranging from radiation–matter interactions to collision physics [1–18]. The search for exact solutions of the problem still deserves attention since the numerical simulations are often affected by accuracy problems and insufficient generality.

In this paper we propose a simple systematic method to obtain exact solutions of the two-level problem based on the dependent-variable transformation and on a general class property of the solutions of the Schrödinger equations. We show that the application of the proposed method allows one to generalize all previously known solutions to more general classes and to derive a variety of new families of integrable models. We present several new analytic solutions of the two-level problem in terms of the hypergeometric function.

The general form of the semiclassical two-level problem is a system of coupled first-order differential equations for probability amplitudes $a_1(t)$ and $a_2(t)$ for the two states $|1\rangle$ and $|2\rangle$, containing two arbitrary real functions of time, $U(t)$ (amplitude modulation; $U > 0$) and $\delta(t)$ (frequency modulation):

$$\mathrm{i}a_{1t} = U\mathrm{e}^{-\mathrm{i}\delta}a_2 \qquad \mathrm{i}a_{2t} = U\mathrm{e}^{+\mathrm{i}\delta}a_1 \tag{1}$$

where the lowercase Latin index denotes differentiation.

This system is equivalent to the following linear second-order ordinary differential equation:

$$a_{1tt} + \left(\mathrm{i}\delta_t - \frac{U_t}{U}\right)a_{1t} + U^2 a_1 = 0. \tag{2}$$

We consider the reduction of this equation to another second-order linear differential equation having a known analytic solution,

$$u_{zz} + f(z)u_z + g(z)u = 0 \tag{3}$$

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via transformation of both independent and dependent variables

$$z = z(t) \tag{4}$$

$$u(z) = \varphi(z) \cdot a_1. \tag{5}$$

The application of this approach [19–21] has recently led to the generalization of all known analytically integrable cases, in terms of confluent hypergeometric functions, to a single formula [20]. Also, several new solvable models, in terms of the Gauss hypergeometric function, were recently derived in [21].

2. The class property

Following our previous papers [19–21], we assume that the function $z(t)$ defining the transformation of the independent variable is a complex-valued function from the real argument t: $z = x(t) + \mathrm{i}y(t)$.

Consider now the formal solutions of the Schrödinger equations *depending on a complex argument* z. As can easily be verified by inspection, if the functions $a^*_{1,2}(z)$ are a solution of the system (1) with this argument for some $U^*(z)$ and $\delta^*(z)$ then the functions $a_{1,2}(t) = a^*_{1,2}(z(t))$ are the solution of (1) for $U(t)$ and $\delta(t)$ given by

$$U(t) = U^*(z)\frac{\mathrm{d}z}{\mathrm{d}t} \qquad \delta_t(t) = \delta^*_z(z)\frac{\mathrm{d}z}{\mathrm{d}t} \tag{6}$$

for an arbitrary complex-valued function $z(t)$. (The last equation of this system is simply another form of the obvious relation $\delta(t) = \delta^*(z(t))$ that, however, is written here in this form for future purposes.) We refer to the pair of functions $U^*(z)$ and $\delta^*(z)$ as the basic integrable model.

It is seen that each *real* basic integrable model of the time-dependent Schrödinger equations generates an entire infinite class of solvable cases of the problem (i.e. real functions $U(t)$ and $\delta(t)$). It is this class property that allows one to generate, via an appropriate choice of the free parameters available, a number of new integrable models with real U and δ. For instance, in the simplest case of constant $U^*(z) = U_0$ and $\delta^*_z(z) = \Delta$ (with arbitrary real U_0 and Δ) one immediately obtains the generalized Rabi class, $U = U_0\,\mathrm{d}z/\mathrm{d}t$ and $\delta_t = \Delta\,\mathrm{d}z/\mathrm{d}t$, presented by Hioe and Carroll [13]. A well known example of an infinite class of solvable models with a non-trivial basic model is the family of Bambini and Berman [10] (for other examples see, for instance, [8–14, 18–21] and references therein). Finally, note that one may try to construct real amplitude and modulation functions $U(t)$ and $\delta(t)$ from complex basic functions $U^*(z)$ and $\delta^*(z)$ using the complex-valuedness of the transformation $z(t)$. Though this fails in general, in many cases this procedure succeeds in producing a generation of new solvable models (see, for instance, [21]).

3. The transformation of the dependent variable

Since the above class property automatically takes into account the independent-variable transformation, we apply only the transformation of the dependent variable (5) that changes equation (3) into the form

$$a_{1zz} + \left(2\frac{\varphi_z}{\varphi} + f\right)a_{1z} + \left(\frac{\varphi_{zz}}{\varphi} + f\frac{\varphi_z}{\varphi} + g\right)a_1 = 0. \tag{7}$$

By comparing this equation with equation (2) rewritten for z, U^*, δ^*, we obtain two nonlinear equations for determination of the functions $U^*(z)$, $\delta^*(z)$ and $\varphi(z)$:

$$\mathrm{i}\delta_z^* - \frac{U_z^*}{U^*} = 2\frac{\varphi_z}{\varphi} + f \qquad U^{*2} = \frac{\varphi_{zz}}{\varphi} + f\frac{\varphi_z}{\varphi} + g. \tag{8}$$

Note that the elimination of $\varphi(z)$ from this system leads to the equation of invariants considered in [19–21]

$$U^{*2} - \frac{1}{2}\left(\mathrm{i}\delta_z^* - \frac{U_z^*}{U^*}\right)_z - \frac{1}{4}\left(\mathrm{i}\delta_z^* - \frac{U_z^*}{U^*}\right)^2 = g - \frac{1}{2}f_z - \frac{1}{4}f^2. \tag{9}$$

However, as will be shown below, the usage of system (8) rather than this equation is more flexible for finding new basic solutions.

In the simplest case of constant φ the system (8) becomes

$$\mathrm{i}\delta_z^* - \frac{U_z^*}{U^*} = f \qquad U^{*2} = g \tag{10}$$

another form of which, derived after exclusion of U^* and application of (6), is well known:

$$U = \sqrt{g}\,\frac{\mathrm{d}z}{\mathrm{d}t} \qquad \mathrm{i}\delta_t = \left(f + \frac{g_z}{2g}\right)\frac{\mathrm{d}z}{\mathrm{d}t}. \tag{11}$$

Indeed, it was this system that was used in [2, 10] to reduce the two-level problem to the hypergeometric equation and in [14] for reduction of the problem to the Riemann–Papperits equation [22], etc. It is seen that, in terms of the class property (6), this system corresponds to the simplest basic solution given by

$$U^* = \sqrt{g} \qquad \delta_z^* = -\mathrm{i}\left(f + \frac{g_z}{2g}\right). \tag{12}$$

However, the system (8) permits a number of other solutions with $\varphi \neq$ constant, as shown for the confluent hypergeometric equation in [20] and for the hypergeometric equation in [21] via reduction of (8) to the equation of invariants.

Consider, for instance, the reduction of the two-level problem to the hypergeometric equation [22, 23],

$$z(1-z)u_{zz} + (Az + B)u_z + Du = 0. \tag{13}$$

The system (8) can then be written as

$$\begin{aligned} \mathrm{i}\delta_z^* - \frac{U_z^*}{U^*} &= 2\frac{\varphi_z}{\varphi} + \frac{Az+B}{z(1-z)} \\ U^{*2} &= \frac{\varphi_{zz}}{\varphi} + \frac{Az+B}{z(1-z)}\frac{\varphi_z}{\varphi} + \frac{D}{z(1-z)}. \end{aligned} \tag{14}$$

Now, trying to find solutions of this equation in the form

$$\frac{\varphi_z}{\varphi} = \frac{\alpha_1}{z} + \frac{\alpha_2}{1-z} \tag{15}$$

$$\frac{U_z^*}{U^*} = \frac{k_1}{z} + \frac{k_2}{1-z} \qquad \delta_z^* = \frac{\delta_1}{z} + \frac{\delta_2}{1-z} \tag{16}$$

we readily find the four independent basic solutions derived in [21]. These basic models are given by

$$U^*/U_0^* = \frac{1}{z(1-z)} \qquad \frac{1}{\sqrt{z}(1-z)} \qquad \frac{1}{1-z} \qquad \frac{1}{\sqrt{z(1-z)}} \tag{17}$$

$$\delta_z^* = \frac{\delta_1}{z} + \frac{\delta_2}{1-z} \tag{18}$$

where the parameters δ_1, δ_2 and U_0^* are arbitrary complex constants. The parameters A, B, C of the hypergeometric equation as well as the parameters $\alpha_{1,2}$ are easily expressed in terms of $\delta_1, \delta_2, U_0^*$ after substitution of (17) and (18) into (14). Further, the application of the class property (6) leads us to four three-parametric classes of integrable models of the two-level problem.

We have checked that the above classes include all presently known analytic solutions of the two-level problem in terms of the hypergeometric function and offer a variety of new solutions.

For example, the basic model $U^* \sim 1/\sqrt{z(1-z)}$ includes the classes of Bambini and Berman [10] (a case with constant detuning, $\delta = \Delta \cdot t$, and real z), Hioe and Carroll [13] (a case with frequency modulation, $\delta(t) \neq \Delta \cdot t$, and real z), Demkov–Kunike [8], etc, as particular subfamilies. Notably, it also contains a new class of solvable models derived by a simple complex-valued choice of z in the form $z = \frac{1}{2} - \mathrm{i}y(t)/2$, given as [21]

$$U(t) = \frac{U_0}{\sqrt{1+y^2}} \frac{\mathrm{d}y}{\mathrm{d}t} \qquad \delta_t(t) = \frac{2\mu y + \lambda}{1+y^2} \frac{\mathrm{d}y}{\mathrm{d}t} \tag{19}$$

where we have set $U_0^* = \mathrm{i}U_0$, $\delta_1 - \delta_2 = 2\mu$, $\delta_1 + \delta_2 = \mathrm{i}\lambda$; U_0, μ, λ are arbitrary real constants. Other examples are the second Demkov–Kunike model [8, 18] (the third basic solution of (17), $z = 1 + \mathrm{e}^{2t/\tau}$) and the class of Carroll and Hioe [14] (the first basic solution of (17), $z = (y + \mathrm{i})/2\mathrm{i}$) derived by mapping equation (2) onto the Riemann–Papperits equation [22] with complex singular points.

Finally, it should be noted that the listed basic models (17) and (18) are not all the possible solutions of system (14). As already mentioned, three other basic models were found recently in [21]. Though the question concerning all possible solutions is at present open, one may try to find more basic solutions of the system (14) starting from different forms of the factor $\varphi(z)$. A motivation for this approach is that, as seen from (14) (or, generally, from (8)), the *a priori* definition of $\varphi(z)$ immediately determines the functions $U^*(z)$ and $\delta^*(z)$. (Then one just has to check whether the latter functions are able to generate real $U(t)$ and $\delta(t)$ when applying (6).) It is for this reason that the usage of system (8) seems to be more convenient than the previously used equation of invariants (9)—a nonlinear equation containing simultaneously two unknown functions. Indeed, below we derive new solutions starting from the following form of φ containing two new parameters, α and p:

$$\varphi = (z+p)^\alpha. \tag{20}$$

4. New solutions

Substitution of equation (20) into (14) immediately yields

$$\frac{U_z^*}{U^*} = \frac{-\frac{1}{2}}{z} + \frac{k}{1-z} + \frac{-1}{p+z} \qquad k = -\tfrac{1}{2}, 0, \tfrac{1}{2} \tag{21}$$

$$\delta_z^* = \frac{\delta_1}{z} + \frac{\delta_2}{1-z} + \frac{\delta_3}{p+z} \qquad 2\alpha = \mathrm{i}\delta_3 + 1 \tag{22}$$

and two additional equations conjugating $\delta_{1,2,3}$, p, α and k.

Thus we have found three more basic solutions

$$U^*/U_0^* = \sqrt{\frac{1-z}{z}}\frac{1}{z+p} \qquad \sqrt{\frac{1}{z}}\frac{1}{z+p} \qquad \sqrt{\frac{1}{z(1-z)}}\frac{1}{z+p} \tag{23}$$

$$\delta_z^* = \frac{\delta_1}{z} + \frac{\delta_2}{1-z} + \frac{\delta_3}{p+z}. \tag{24}$$

Consider now the constant-detuning case, $\delta(t) = \Delta \cdot t \Leftrightarrow \delta_t = \Delta = \text{constant}$.

The second equation of (6) and equation (22) define the following relation between t and z:

$$\mathrm{e}^t = \frac{z^\mu (z+p)^\nu}{(1-z)^{\lambda+\mu}} \tag{25}$$

where we have set $\delta_1 = \Delta\mu$, $\delta_2 = \Delta(\lambda+\mu)$ and $\delta_3 = \Delta\nu$. Then the first equation of (6) gives the following three new classes of pulses:

$$U(t) = \frac{U_0\sqrt{z}(1-z)^{1+k}}{(\lambda z+\mu)(z+p)+\nu z(1-z)} \qquad k = -\tfrac{1}{2}, 0, \tfrac{1}{2}. \tag{26}$$

The derived classes are three-parametric since five parameters $\mu, \lambda, \nu, U_0, p$ are conjugated by two additional relations given as

$$\nu = \frac{2(\lambda p - \mu)}{1+(1-2k)p} \tag{27}$$

$$\frac{p}{(1+p)^{2k}} = \frac{4U_0^2}{1+\Delta^2\nu^2}. \tag{28}$$

At an appropriate choice of the parameters, the transformation (25), (27), (28) defines a one-to-one mapping of the axis t onto the segment $z \in [0, 1]$ (see figure 1). Then the pulses of the classes (26) are bell-shaped asymmetric functions vanishing at $t \to \pm\infty$. The pulse shapes are shown in figure 2. It is seen that the shapes are analogous to the well known shapes treated by Bambini–Berman [10].

The solution of the two-level problem with the initial condition $a_1(-\infty) = 1$, $a_2(-\infty) = 0$, is explicitly given by

$$a_1 = \left(1+\frac{z}{p}\right)^{-(1+\mathrm{i}\Delta\nu)/2} {}_2F_1\,(a, b; c;\ z) \tag{29}$$

$$-(a+b+1) = A = -\tfrac{1}{2} + k + \mathrm{i}\Delta\lambda \tag{30}$$

$$c = B = \tfrac{1}{2} + \mathrm{i}\Delta\mu \tag{31}$$

$$-ab = D = \frac{U_0^2}{p^2} - \left(\tfrac{1}{2} + \mathrm{i}\Delta\mu\right)\frac{1+\mathrm{i}\Delta\nu}{2p} \tag{32}$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function [23]. It is seen that the class with $k = -\frac{1}{2}$ is rather similar to the class of Bambini–Berman. Indeed, the parameters A and B are the same (see [10], formulae (15*a*) and (15*b*)) and D, given by (32), at $p = 1$ differs from that of Bambini–Berman only by the second term.

The occupation probability $W_1 = |a_1|^2$ versus z is shown in figure 3. For comparison, we present in figure 4 the probabilities for the corresponding Bambini–Berman pulse.

The final probability after the interaction is given by

$$W_1(+\infty) = \frac{p}{1+p}\frac{\pi\,\mathrm{sech}(\pi\Delta\mu)\,|\Gamma(1+k+\mathrm{i}\Delta(\lambda+\mu))|^2}{\left|\Gamma\left(\frac{1}{2}+\mathrm{i}\Delta\mu-a\right)\Gamma\left(\frac{1}{2}+\mathrm{i}\Delta\mu-b\right)\right|^2}. \tag{33}$$

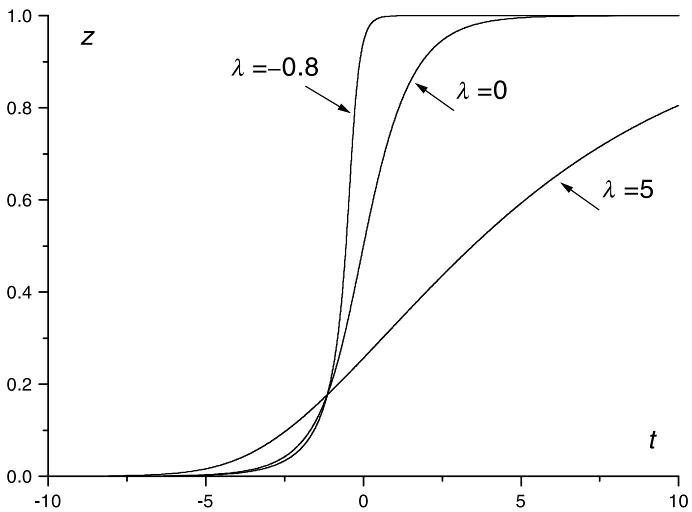

Figure 1. The independent-variable transformation given by equation (25), $k = -\frac{1}{2}$, $\Delta = \mu = 1$, $p = \frac{1}{2}$.

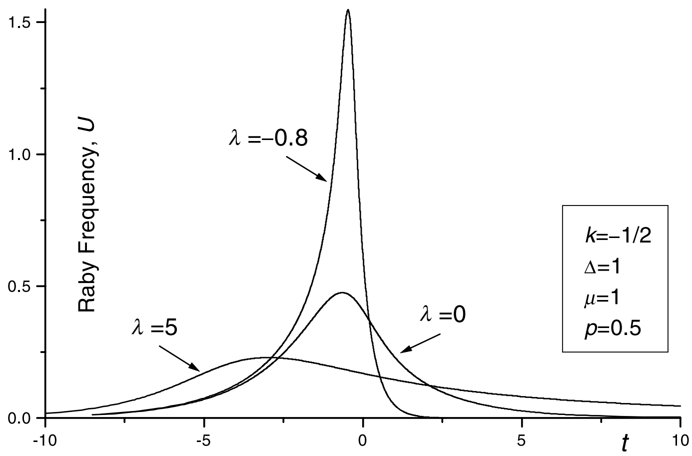

Figure 2. Pulse shapes given by equation (26), $k = -\frac{1}{2}$, $\Delta = \mu = 1$, $p = \frac{1}{2}$.

While the Euler gamma function in the numerator of this expression is always (when $k = -\frac{1}{2}, 0, \frac{1}{2}$) written in terms of elementary functions, the gamma functions in the denominator, in general, are not. However, in certain cases it is possible to obtain simple expressions in terms of elementary functions. For instance, it is the case when $\mathrm{Re}(a)$ and $\mathrm{Re}(b)$ are integer or half-integer (see [23]). In particular, when $p = \mu/\lambda$ ($\Rightarrow \nu = 0, U_0 = p^{1/2}(1+p)^{-k}/2$) one can check that $\mathrm{Re}(a) = -\frac{1}{2}$, $\mathrm{Re}(b) = -k$, so that the final occupation probability is written as

$$W_1(+\infty) = \begin{cases} \tanh(\pi\Delta\mu)\tanh(\pi\Delta(\lambda+\mu)) & k = -\frac{1}{2} \\ \tanh(\pi\Delta\mu)\coth(\pi\Delta(\lambda+\mu)) & k = 0 \\ \mathrm{sech}(\pi\Delta\mu)\,\mathrm{sech}(\pi\Delta(\lambda+\mu)) & \\ \quad \times \sinh(\pi\Delta\mu - \pi\,\mathrm{Im}(a))\sinh(\pi\Delta\mu - \pi\,\mathrm{Im}(b)) & k = \frac{1}{2} \end{cases} \tag{34}$$

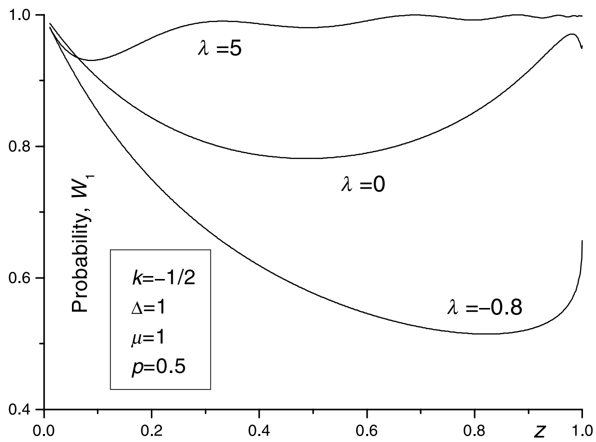

Figure 3. Occupation probability for the first level at $k = -\frac{1}{2}$, $\Delta = \mu = 1$, $p = \frac{1}{2}$.

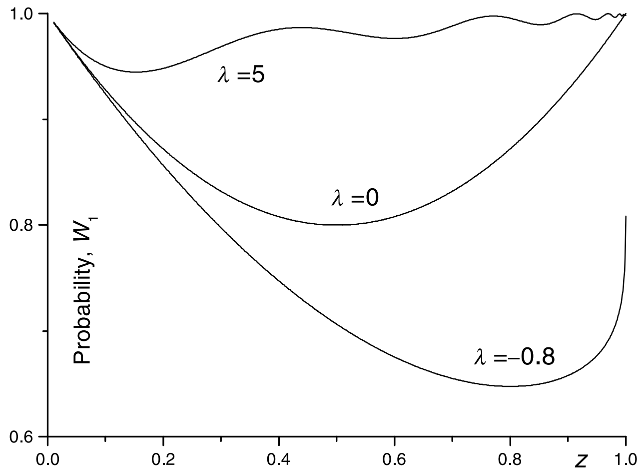

Figure 4. First-level occupation probability for the Bambini–Berman pulse $\Delta = \mu = 1$, $U_0 = 1$.

where

$$\mathrm{Im}(a), \mathrm{Im}(b) = -\tfrac{1}{2}\Delta\lambda\left(1 \pm \sqrt{1 - \frac{\mu}{\Delta^2\lambda^2(\lambda+\mu)}}\right). \tag{35}$$

5. Summary

We have introduced a simple technique to find analytically integrable cases of the two-level problem. The approach is based on the class property (6) of the solutions of the time-dependent Schrödinger equations (combined with a complex-valued, in general, transformation of the independent variable) and on the dependent-variable change. The procedure for finding the basic integrable models consists in the direct mapping of the initial equation (2) onto some standard second-order linear ordinary differential equation preliminarily changed by means of a transformation of the dependent variable. Due to the complex-valuedness of the independent-variable transformation the complex basic models may also be used for construction of new solvable cases of the two-level problem. The approach allows one to derive all the presently known solvable cases, and to generate a number of new classes of analytically integrable models. We have presented several new families solvable in terms of the Gauss hypergeometric function.

Finally, we would like to note that the proposed technique, as it is a systematic one, can be applied to other structurally analogous problems. For instance, the extension of the approach to the three-level problem is straightforward [24].

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